PROCESSES, FUNCTIONS, APPLICATION & COMPOSITION

1. Processes

The term **function** seems to suggest an action, behavior, or process capable of producing a result. Given a definition of **process**, a function could be considered a special process.

Definition 1.1. Process: Two sets $\mathbf{f}$ and σ define a process $\mathbf{f}_{(\sigma)}$ iff

$$(\exists x)\big(\ \mathbf{f}_{(\sigma)}(x) \neq \varnothing\big)\ \&\ (\forall \mathbf{g})(\mathbf{g} \subseteq \mathbf{f})(\exists x)\big(\ \mathbf{g}_{(\sigma)}(x) \neq \varnothing\ \big).$$

[Where '$\subseteq$' means non-empty subset.]

The above definition only asserts that a process is an abstraction somehow related to an interaction of sets. No demands are made on how such a relationship need be supported. The important property of a process is that it is not a set. '$\mathbf{f}_{(\sigma)}(x) \in_s \mathbf{Q}$' makes mathematical sense, while '$\mathbf{f}_{(\sigma)} \in_s \mathbf{Q}$' does not. The expression '$\mathbf{f}_{(\sigma)}(x)$' defines a set-membership condition, while the expression '$\mathbf{f}_{(\sigma)}$' defines a set-behavior.

Definition 1.2. Process Equality:

$$\mathbf{f}_{(\sigma)} = \mathbf{g}_{(\omega)} \iff (\forall x)(\ \mathbf{f}_{(\sigma)}(x) = \mathbf{g}_{(\omega)}(x)\ ).$$

All uses of processes are only a prediction, the actual set behavior can not be realized until a process is given a set-theoretic interpretation.

2. Applications

Given that extended image operation, $\mathbf{f}\big[x\big]_{\sigma}$, has a well defined set-theoretic definition, the following definitions of *application* transform a process into a set-theoretic reality. Since a process could be applied to a set or to a process, two definitions of application are required.

Definition 2.1. Application(set): $\mathbf{f}_{(\sigma)}(x) \ = \ \mathbf{f}\big[x\big]_{\sigma}$.

A process is a function if the image of every singleton produces a singleton set.

Definition 2.2. Function:

$$\mathbf{f}_{(\sigma)} \text{ is a function } \iff (\forall y)\Big(\ \mathit{Sing}(y)\ \ \&\ \ \mathbf{f}\big[y\big]_{\sigma} \neq \varnothing\ \rightarrow\ \ \mathit{Sing}\Big(\mathbf{f}\big[y\big]_{\sigma}\Big)\ \Big).$$

Note: This definition of function does not require nor limit a function to being a graph.

Definition 2.3. Application(process): $\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)}) = \Big(\mathbf{f}_{(\sigma)}(\mathbf{g})\Big)_{(\omega)} = \Big(\mathbf{f}\big[\mathbf{g}\big]_{\sigma}\Big)_{(\omega)}$.

A process applied to a set produces a set. When applied to a process produces a process.

3. Nested Applications

Though application is well defined, sequences of applications are not. Consider the simple expression $\mathbf{f}_{(\sigma)}\mathbf{g}_{(\omega)}(x)$. Without proper bracketing or an explicitly defined bracketing convention, its meaning is ambiguous. There are two legitimate interpretations: $\mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(x)\Big)$ and $\Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(x)$. The differences are explicit in the following examples.

Example 3.1. *Interpretations of* $\mathbf{f}_{(\sigma)}\mathbf{g}_{(\omega)}(x)$.

$$(a)\ \ \mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(x)\Big) \ = \ \mathbf{f}\big[\mathbf{g}\big[x\big]_{\omega}\big]_{\sigma},$$

$$(b)\ \ \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(x) \ = \ \Big(\mathbf{f}_{(\sigma)}(\mathbf{g})\Big)_{(\omega)}(x) \ = \ \Big(\mathbf{f}\big[\mathbf{g}\big]_{\sigma}\Big)\big[x\big]_{\omega}.$$

Example 3.2. *Interpretations of* $\mathbf{f}_{(\sigma)}\mathbf{g}_{(\omega)}\mathbf{h}_{(\tau)}(x)$.

$$
\begin{array}{ll}
(a) & \mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}\Big(\mathbf{h}_{(\tau)}(x)\Big)\Big) \;=\; \mathbf{f}\Big[\mathbf{g}\big[\mathbf{h}[x]_{\tau}\big]_{\omega}\Big]_{\sigma}, \\
(b) & \mathbf{f}_{(\sigma)}\Big(\big(\mathbf{g}_{(\omega)}(\mathbf{h}_{(\tau)})\big)(x)\Big) \;=\; \mathbf{f}_{(\sigma)}\Big(\big(\mathbf{g}_{(\omega)}(\mathbf{h})\big)_{(\tau)}(x)\Big) \;=\; \mathbf{f}\Big[\Big(\mathbf{g}\big[\mathbf{h}\big]_{\omega}\Big)\big[x\big]_{\tau}\Big]_{\sigma}, \\
(c) & \mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(\mathbf{h}_{(\tau)})\Big)(x) \;=\; \mathbf{f}_{(\sigma)}\Big(\big(\mathbf{g}_{(\omega)}(\mathbf{h})\big)_{(\tau)}\Big)(x) \;=\; \mathbf{f}\Big[\mathbf{g}\big[\mathbf{h}\big]_{\omega}\Big]_{\sigma}\big[x\big]_{\tau}, \\
(d) & \Big(\mathbf{f}_{(\sigma)}\big(\mathbf{g}_{(\omega)}\big)\big(\mathbf{h}_{(\tau)}\big)\Big)(x) \;=\; \Big(\big(\mathbf{f}_{(\sigma)}(\mathbf{g})\big)_{(\omega)}(\mathbf{h})\Big)_{(\tau)}(x) \;=\; \Big(\mathbf{f}\big[\mathbf{g}\big]_{\sigma}\big[\mathbf{h}\big]_{\omega}\Big)\big[x\big]_{\tau}, \\
(e) & \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)\Big(\mathbf{h}_{(\tau)}(x)\Big) \;=\; \Big(\mathbf{f}_{(\sigma)}(\mathbf{g})\Big)_{(\omega)}\Big(\mathbf{h}_{(\tau)}(x)\Big) \;=\; \mathbf{f}\big[\mathbf{g}\big]_{\sigma}\big[\mathbf{h}[x]_{\tau}\big]_{\omega}.
\end{array}
$$

Note: Interpretations for sequences greater than three gets large rather quickly with 14 for four and 42 for five.

It may not be immediately apparent that even the simplest case has more that one valid interpretation. Therefore it must be shown that there is a case where both interpretations are non-empty and not equal to each other,

Example 3.3. *Present a case such that:*

$$\mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(\mathbf{X})\Big) \neq \varnothing, \quad \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(\mathbf{X}) \neq \varnothing \quad \textit{and} \quad \mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(\mathbf{X})\Big) \neq \Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(\mathbf{X}).$$

Let: $\mathbf{f} = \{<\mathbf{y},\mathbf{z}>,<\mathbf{a},\mathbf{x},\mathbf{b},\mathbf{k}>\}$ *and* $\mathbf{g} = \{<\mathbf{x},\mathbf{y}>,<\mathbf{a},\mathbf{b}>\}$ *with* $\sigma = \, << 1,3>,< 2,4>>$ $\omega = \, <<1>,<2>>$.

Then: $\mathbf{f}_{(\sigma)}\Big(\mathbf{g}_{(\omega)}(\{<\mathbf{x}>\})\Big) = \{<\mathbf{z}>\}$ *and* $\Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big)(\{<\mathbf{x}>\}) = \{<\mathbf{k}>\}$

Note: $\Big(\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)})\Big) = \{<\mathbf{x},\mathbf{k}>\}_{(\omega)}$ *and* $\{<\mathbf{x},\mathbf{k}>\}_{(\omega)}(\{<\mathbf{x}>\}) = \{<\mathbf{k}>\}$ *also*

$\mathbf{g}_{(\omega)}(\{<\mathbf{x}>\}) = \{<\mathbf{y}>\}$ *and* $\mathbf{f}_{(\sigma)}\Big(\{<\mathbf{y}>\}\Big) = \{<\mathbf{z}>\}$.

4. Self Application

Self application can be difficult to formally model. Such is not the case under XST.[1]

Example 4.1. *Let* $\mathbf{f} = \{< a,b >\}$ & $\mathbf{g} = \{< b,a >\}$ *with* $\sigma = \, << 1 >,< 2,1 >>$,
then $\mathbf{f}_{(\sigma)}(\mathbf{f}_{(\sigma)}) = \mathbf{g}_{(\sigma)}$ *and* $\mathbf{g}_{(\sigma)}(\mathbf{g}_{(\sigma)}) = \mathbf{f}_{(\sigma)}$

Example 4.2. *Let* $\mathbf{S} = \{< x^i, x^{2i} >: i \in \mathbf{I} \;\&\; x \in \mathbf{N}\}$, *with* $\sigma = \, << 1 >,< 2 >>$,
then $\mathbf{S}_{(\sigma)}(\mathbf{S}_{(\sigma)}) \;=\; \{< x^i, x^{4i} >: i \in \mathbf{I} \;\&\; x \in \mathbf{N}\}_{(\sigma)}$.
giving $\mathbf{S}_{(\sigma)}(\{< 2 >\}) = (\{< 4 >\})$ & $\mathbf{S}_{(\sigma)}(\mathbf{S}_{(\sigma)})(\{< 2 >\}) = \{< 16 >\}$.

5. Lambda Application

Both the Lambda abstraction, $\lambda(x).t$, and the XST expression, $\mathbf{f}_{(\sigma)}$, define the *predicted behavior* of a *computable process* under an application.

Given $\lambda(\mathbf{v}).\mathbf{t} = \{<\mathbf{v},\mathbf{w}>: \mathbf{t} = \mathbf{w}\}$, where $\mathbf{v}$ and $\mathbf{w}$ are distict variables and $\mathbf{w}$ does not occur in the term $\mathbf{t}$,[2](p. 90)
Then, for $\mathbf{f} = \{<\mathbf{v},\mathbf{w}>:<\mathbf{t}>=<\mathbf{w}>\}$, $\lambda(\mathbf{v}).\mathbf{t} = \mathbf{f}_{(\sigma)}(<\mathbf{t}>)$, $\sigma = \, << 1 >,< 2 >>$.

For $x \mapsto x^2 + 2$ gives:
In Lambda: $\lambda(x).x^2 + 2$ for $\mathbf{f}(x) = x^2 + 2$ then $\mathbf{f}(5) = 27$
In XST: $\mathbf{f} = \{< x,z >: z = x^2 + 2\}$ for $\sigma = \, << 1 >,< 2 >>$ $\mathbf{f}_{\sigma}(\{< 5 >\}) = \{< 27 >\}$

For $x \mapsto \sqrt{x}$ gives:
In Lambda: $\lambda(x).\sqrt{x}$ for $\mathbf{f}(x) = \sqrt{x}$ then $\mathbf{f}(4) = 2$ and $\mathbf{f}(\mathbf{f}(16)) = 2$
In XST: $\mathbf{f} = \{< x,y,z >: y = |\sqrt{x}|, z = -|\sqrt{x}|\}$
for $\sigma = \, << 1 >,< 2 >>$ $\mathbf{f}_{\sigma}(\{< 4 >\}) = \{< 2 >\}$
for $\sigma = \, << 1 >,< 3 >>$ $\mathbf{f}_{\sigma}(\{< 4 >\}) = \{< -2 >\}$
for $\sigma = \, << 1 >,< 3 >>$ $\mathbf{f}_{\sigma}(\mathbf{f}_{\sigma}(\{< 16 >\})) = \{< -2i >\}$

A Lambda application is restricted to evaluating processes defined by a graph or a subset of some Cartesian product. An XST application can evaluate any well-defined XST process.

6. Composition

Composition is an act of aggregating the interactive resultant behavior of multiple processes (functions) into a single process (function). Besides its categorical relevance for studying equivalent system behaviors, it can be used constructively to produce alternative computer programs by eliminating the intermediate operations indirectly contributing to resultant behavior. The execution of operation, **h**, in the diagram below is equivalent to executing **f** followed by an execution of **g**.

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "\mathbf{f}_{(\sigma)}"] \arrow[dr, "\mathbf{h}_{(\tau)}"'] & \mathbf{B} \arrow[d, "\mathbf{g}_{(\omega)}"'] \\
 & \mathbf{C}
\end{tikzcd}$$

Two problems arise. It is not obvious that **h** can always be defined in terms of a given **f** and **g**, and even if it can be, it may not be of any value since processes are abstract mathematical objects with no authority to execute on a computer.

It needs to be shown that given any two processes (functions) that are compositable there is always a constructible composition and that the resultant process (function) is definable in terms of structured sets.

Definition 6.1. Composition: $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} = \Big(\mathbf{f} \Big/^{<\omega_1,\ \omega_2>}_{<\sigma_1,\ \sigma_2>} \mathbf{g} \Big)_{(<\sigma_1,\ \omega_2>)}$.

Theorem 6.2. *If* $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}$ *exists there exists* **h** *and* τ *such that* $\mathbf{h}_{(\tau)} = \mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)}$.

Proof:

1) Given $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} = \Big(\mathbf{f} \Big/^{<\omega_1,\ \omega_2>}_{<\sigma_1,\ \sigma_2>} \mathbf{g} \Big)_{(<\sigma_1,\ \omega_2>)}$, let $\mathbf{h} = \mathbf{f} \Big/^{<\omega_1,\omega_2>}_{<\sigma_1,\sigma_2>} \mathbf{g}$

2) For $\tau = <\sigma_1, \omega_2>$, $\mathbf{h} = \mathbf{f} \Big/^{<\omega_1,\tau_2>}_{<\tau_1,\sigma_2>} \mathbf{g}$

3) then $\mathbf{g}_{(\omega)} \circ \mathbf{f}_{(\sigma)} = \Big(\mathbf{f} \Big/^{<\omega_1,\ \tau_2>}_{<\tau_1,\ \sigma_2>} \mathbf{g} \Big)_{(\tau)} = \mathbf{h}_{(\tau)}$.

References

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[2] Suppes, P.:*Axiomatic Set Theory*, Van Nostrand, 1960
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Appendix A. XST Definitions [3]

Definition A.1. Re-Scope by Scope:

$$\mathbf{z}^{/\sigma/} = \big\{\mathbf{x}^{w} : \ (\sigma = \varnothing \ \& \ \mathbf{x} \in_{w} \sigma) \ \mathit{or} \ (\exists s)\big(\ \mathbf{x} \in_{s} \mathbf{z} \ \ \& \ \ s \in_{w} \sigma \ \big)\big\}.$$

$$\textit{examples } \{.., \mathbf{x}^{s}, ..\}^{/\{..,s^{w},..\}/} = \{.., \mathbf{x}^{w}, ..\}, \ \ \mathbf{z}^{/\varnothing/} = \varnothing.$$

Definition A.2. Re-Scope by Element:

$$\mathbf{z}^{\backslash\sigma\backslash} = \big\{\mathbf{x}^{\mathbf{s}} : \ (\sigma = \varnothing \ \& \ \mathbf{x} \in_{s} \mathbf{z}) \ \mathit{or} \ (\exists w)\big(\ \mathbf{x} \in_{w} \mathbf{z} \ \ \& \ \ s \in_{w} \sigma \ \big)\big\}.$$

$$\textit{examples } \{.., \mathbf{x}^{w}, ..\}^{\backslash\{..,s^{w},..\}\backslash} = \{.., \mathbf{x}^{s}, ..\} \ \ \mathbf{z}^{\backslash\varnothing\backslash} = \mathbf{z}.$$

These two definitions are duals. They both replace scope values of a given set. One with scope values from another set. The other with element values from another set.

For example:

$$\{x^a, y^b, z^d\}^{/\{a^A, b^B, d^D\}/} = \{x^A, y^B, z^D\},$$
$$\{x^A, y^B, z^D\}^{\backslash\{a^A, b^B, d^D\}\backslash} = \{x^a, y^b, z^d\}.$$

Definition A.3. n-Tuple: $< x_1, x_2, .., x_n > = \{x_1^1, x_2^2, .., x_n^n\}$.

Definition A.4. Domain Extraction:

$$\text{Đ}_\sigma(\mathbf{Q}) = \Big\{ \mathbf{x}^s : (\exists \mathbf{z}, w)\big(\mathbf{z} \in_w \mathbf{Q}\big) \ \ \& \ \ \big[\big(\sigma = \varnothing \ \& \ \mathbf{x} = \mathbf{z} \ \& \ s = w\big) \ \mathit{or} \ \big(\mathbf{x} = \mathbf{z}^{/\sigma/} \neq \varnothing \ \ \& \ s = w^{/\sigma/} \ \big)\big]\Big\}.$$

For example:

$$\text{Đ}_{\{A^1, C^2\}}(\{\{a^A, b^B, c^C\}\}) = \{\{a^1, c^2\}\},$$
$$\text{Đ}_{<3,1>}(\{\{a^1, b^2, c^3\}^{\{A^1, B^2, C^3\}}\}) = \{< c, a >^{<C,A>}\},$$
$$\text{Đ}_{\{3^1, 1^2, y^9, v^5, v^7, Q^A\}}(\{\{a^1, b^2, c^3\}^{\{x^y, w^v, z^Q\}}\}) = \{< c, a >^{\{x^9, w^5, w^7, z^A\}}\}.$$

Definition A.5. Restriction:

$$\mathbf{Q}\,|_{\sigma}\,\mathbf{A} = \Big\{ \mathbf{z}^w \colon \big(\mathbf{z} \in_w \mathbf{Q}\big) \ \& \ \ \big(\exists\ a, s\big)\big(a \in_s \mathbf{A} \ \& \ a^{\backslash\sigma\backslash} \dot{\subseteq} \mathbf{z} \ \ \& \ s^{\backslash\sigma\backslash} \subseteq w\big)\Big\}.$$

$\mathbf{Q}\,|_{\sigma}\,\mathbf{A}$ yields a subset of $\mathbf{Q}$ *restricted by* $\mathbf{A}$ under the influence of σ. This provides a wide variety for functionally selecting related items from elements of the set $\mathbf{Q}$.

Definition A.6. Image: $\mathbf{Q}\big[\mathbf{A}\big]_{<\sigma_1, \sigma_2>} = \text{Đ}_{\sigma_2}(\mathbf{Q}\,|_{\sigma_1}\,\mathbf{A})$.

Definition A.7. Image:(alternate definition)

$$\mathbf{Q}\big[\mathbf{A}\big]_{<\sigma_1, \sigma_2>} = \Big\{ \mathbf{x}^{\mathbf{s}} \colon \big(\exists\ z, w, a, t\big)\big(\ z \in_w \mathbf{Q}\big) \ \ \& \ \ \big(a \in_t \mathbf{A}\big) \ \ \& \ \ 1) \ \big(a^{\backslash\sigma_1\backslash} \dot{\subseteq} z \ \ \& \ t^{\backslash\sigma_1\backslash} \subseteq w\big) \ \ \& \ \ 2) \ \big(\mathbf{x} = z^{/\sigma_2/} \neq \varnothing \ \ \& \ \mathbf{s} = w^{/\sigma_2/} \ \big)\Big\}.$$

[Where '$\dot{\subseteq}$' means non-empty subset.]

Appendix B. Relative Product

Relative Product has more personality than other operations in that given the same two operands the resultant set can have many forms. Though, in CST the operation is rather bland matching the range elements of the first operand with the domain elements of the second operand and producing a pair of the domain element of the first with the range element of the second.

For example, in CST: $\{< a, b >\}/\{< b, c >\} = \{< a, c >\}$. Following are some element combinations that are potentially more interesting:

1) $< a, b > \ \ \& < b, c > \longrightarrow < a, c >$,
2) $< a, b > \ \ \& < b, c > \longrightarrow < a, b, c >$,
3) $< a, b > \ \ \& < a, c > \longrightarrow < a, b, c >$,
4) $< a, b > \ \ \& < a, c > \longrightarrow < b, c >$,
5) $< a, c > \ \ \& < b, c > \longrightarrow < a, b, c >$,
6) $< a, c > \ \ \& < b, c > \longrightarrow < a, b >$,
7) $< a, b, c > \ \ \& < x, y, c, b > \longrightarrow < b, c, a, y, b, c, x, x >$,
8) $< a, b, c > \ \ \& < x, y, z > \longrightarrow < a, b, c, x, y, z >$.

All of the above are producible with the following definition.

Definition B.1. Relative Product:

$$\mathbf{F}\Big/^{<\omega_1, \omega_2>}_{<\sigma_1, \sigma_2>}\mathbf{G} = \Big\{\mathbf{z}^\tau \colon (\exists x, s, y, t)\big(\ x \in_s \mathbf{F} \ \& \ y \in_t \mathbf{G} \ \ \& \ \ x^{/\sigma_2/} = y^{/\omega_1/} \ \ \& \ \ s^{/\sigma_2/} = t^{/\omega_1/} \ \ \& \ \ \mathbf{z} = x^{/\sigma_1/} \cup y^{/\omega_2/} \ \ \& \ \ \tau = s^{/\sigma_1/} \cup t^{/\omega_2/}\big) \ \Big\}.$$

For $\mathbf{f}\Big/^{\omega}_{\sigma}\mathbf{g}$ using element combinations (1)-(8) above, the following values for σ and ω support the corresponding mappings above.

1) $\sigma =< \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega =< \{1^1\},\ \{2^2\} >,$
2) $\sigma =< \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega =< \{1^1\},\ \{1^2, 2^3\} >,$
3) $\sigma =< \{1^1, 2^2\},\ \{1^1\} > \ \ \&\ \ \omega =< \{1^1\},\ \{2^3\} >,$
4) $\sigma =< \{2^1\},\ \{1^1\} > \ \ \&\ \ \omega =< \{1^1\},\ \{2^2\} >,$
5) $\sigma =< \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega =< \{2^1\},\ \{1^2, 2^3\} >,$
6) $\sigma =< \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega =< \{2^1\},\ \{1^2\} >,$
7) $\sigma =< \{2^1, 3^2, 1^3\}, \{2^1, 3^2\} > \ \ \&\ \ \omega =< \{4^1, 3^2\}\{2^4, 4^5, 3^6, 1^7, 1^8\} >,$
8) $\sigma =< \{1^1, 2^2, 3^3\},\ \varnothing > \ \ \&\ \ \omega =< \varnothing,\ \{4^1, 5^2, 6^3\} > .$

Appendix C. XST & CST Functions

XST functions are defined by *set behavior* and not as sets of ordered pairs, as are CST functions. When functions are defined as specific sets (graphs, or subsets of Cartesian products) difficulties may arise when trying to model general systems of functions.

If a definition of function, radically different than any established CST definition is to be tolerated, it must not only remove perceived CST difficulties but also preserve existing dependencies on CST defined functions.

One *CST difficulty* encountered when modeling the behavior of a function applied to a function is called the **CST cardinality difficulty**. If a set contains at least two elements, then the set of CST functions from that set to itself has a greater cardinality than the set itself. Whatever conclusions are drawn from this statement do not hold for XST functions, since a construction exists that allows the set of all functions from any set of cardinality **n** to itself to have cardinality **n**.

Given a set $\mathbf{Q} = \{< a >, < b >, < c >\}$ a set of cardinality 3 can be defined that supports all 27 functions from $\mathbf{Q}$ to $\mathbf{Q}$. For $\mathbf{F}$ defined as follows:

$$\begin{aligned}\mathbf{F} = \{&< \mathbf{a}\ \ \mathbf{a,a,a,a,a,a,a,a,a,}\ \ \mathbf{b,b,b,b,b,b,b,b,b,}\ \ \mathbf{c,c,c,c,c,c,c,c,c} >,\\ &< \mathbf{b}\ \ \mathbf{a,a,a,b,b,b,c,c,c,}\ \ \mathbf{a,a,a,b,b,b,c,c,c,}\ \ \mathbf{a,a,a,b,b,b,c,c,c,} >,\\ &< \mathbf{c}\ \ \mathbf{a,b,c,a,b,c,a,b,c,}\ \ \mathbf{a,b,c,a,b,c,a,b,c,}\ \ \mathbf{a,b,c,a,b,c,a,b,c} >\ \ \}\end{aligned}$$

For $\mathbf{F}_{(\tau)}(\mathbf{F}_{(\sigma)}) = \mathbf{f}^{(i)}$ and the proper choices for τ and σ, any of the 27 CST functions can be derived. Where $\mathbf{f}(\mathbf{x}) = \{y :< x, y > \in \mathbf{f}\}$

Let $\tau =<< 1, ..., 28 >, < 1, i + 1 >>$ and $\sigma =<< 1 >, < 2 >>$.

Then $\mathbf{F}_{(\tau)}(\mathbf{F}_{(\sigma)}) = \mathbf{f}^{(i)}$ with

$\mathbf{f}^{(\mathbf{6})} = \{< \mathbf{a}, \mathbf{a} >, < \mathbf{b}, \mathbf{b} >, < \mathbf{c}, \mathbf{c} >\}$
$\mathbf{f}^{(\mathbf{22})} = \{< \mathbf{a}, \mathbf{c} >, < \mathbf{b}, \mathbf{b} >, < \mathbf{c}, \mathbf{a} >\}$
$\mathbf{f}^{(\mathbf{1})} = \{< \mathbf{a}, \mathbf{a} >, < \mathbf{b}, \mathbf{a} >, < \mathbf{c}, \mathbf{a} >\}$
$\mathbf{f}^{(\mathbf{14}} = \{< \mathbf{a}, \mathbf{b} >, < \mathbf{b}, \mathbf{b} >, < \mathbf{c}, \mathbf{b} >\}$
$\mathbf{f}^{(\mathbf{27})} = \{< \mathbf{a}, \mathbf{c} >, < \mathbf{b}, \mathbf{c} >, < \mathbf{c}, \mathbf{c} >\}$

$\textbf{Đ}_{\sigma}(\mathbf{F}) = \{\ < \mathbf{a}, \mathbf{a} >, < \mathbf{b}, \mathbf{c} >, < \mathbf{c}, \mathbf{b} >\ \},\ \sigma =< \mathbf{1}, \mathbf{9} >$

A function space of all CST functions from set **A** to set **B** can be defined by:

$$\mathbf{B}^{\mathbf{A}}\ =\ \{\mathbf{f} : \mathbf{f}\ \textit{is a}\ \mathbf{CST}\ \textit{function}\ \ \&\ \ \textbf{Đ}_{<1>}(\mathbf{f}) = \mathbf{A}\ \&\ \ \textbf{Đ}_{<2>}(\mathbf{f}) \mathrel{\underline{\subset\!\cdot}} \mathbf{B}\ \}.$$

An equivalence class of all tuple-constructions from set **A** to set **B** can be defined by:

$$\mathbf{Fs}(\mathbf{A}, \mathbf{B}) = \{\mathbf{F} : (\forall\ \mathbf{f})(\mathbf{f} \in \mathbf{B}^{\mathbf{A}})(\exists \mathbf{i})(\sigma =< \mathbf{1}, \mathbf{i} > \&\ \ \textbf{Đ}_{\sigma}(\mathbf{F}) = \mathbf{f}\ \}.$$

For a non self mapping example, let $\mathbf{A} = \{< a >, < b >\}$ and $\mathbf{B} = \{< x >, < y >, < z >\}$, a set, $\mathbf{G}$, of cardinality 2 can be defined that supports all 9 functions from **A** to **B**.

$$\begin{aligned}\mathbf{G} = \{&< \mathbf{a}\ \ \mathbf{x.x.x.}\ \ \mathbf{y,y,y,}\ \ \mathbf{z,z,z} >,\\ &< \mathbf{b}\ \ \mathbf{x,y,z,}\ \ \mathbf{x,y,z,}\ \ \mathbf{x,y,z} >\ \}\end{aligned}$$