XST DEFINITIONS, OPERATIONS, AND PROPERTIES - TUPLES, GRAPHS, FUNCTIONS -

1. INTRODUCTION

Extended set theory, XST, differs from Classical set theory, CST, by assuming a ternary membership condition instead of a binary membership condition. Where the membership condition of CST establishes $\mathbf{x}$ to be an element of a set $\mathbf{A}$ whenever the truth-functional $\Gamma_A(\mathbf{x})$ is true, an XST membership condition establishes $\mathbf{x}$ to be a $\mathbf{y}$-element of a set $\mathbf{A}$ whenever the truth-functional $\Gamma_A(\mathbf{x}, \mathbf{y})$ is true. The $\mathbf{y}$ qualifier is called the *scope* component of XST set membership. XST axioms and notation are fully presented in [1].

2. XST Basic Definitions

Definition 2.1. Set: Given the null set, $\varnothing$, and the extended membership predicate, $\in$,

$$Y \text{ is a set } \longleftrightarrow (\exists x, s)(\ x \in_s Y\ ) \text{ or } Y = \varnothing, .$$

Notationally, XST sets are written as in CST except with subscripts on *element of*, $\in_s$, in conditional statements, and with superscripts on elements in extensional descriptions, $\{\ x^a, y^b, z^c\ \}$. A set qualifies as a CST set iff all its elements are either individuals or other CST sets, all having null scope values: $\{a, b, \{c\}\} \equiv \{a^\varnothing, b^\varnothing, \{c^\varnothing\}^\varnothing\}$.

Definition 2.2. Scope Set: $\mathcal{S}(\mathbf{A}) = \{y^\varnothing\ (\exists x)(\ x \in_y \mathbf{A}\ )\}$.

EXAMPLE: $\mathcal{S}(\{a^A, b^B, c^C\}) = \{A^\varnothing, B^\varnothing, C^\varnothing\} = \{A, B, C\}$.

Definition 2.3. Element Set: $\mathcal{E}(\mathbf{A}) = \{\ x^\varnothing\ (\exists y)(\ x \in_y \mathbf{A})\}$.

EXAMPLE: $\mathcal{E}(\{a^A, b^B, c^C\}) = \{a^\varnothing, b^\varnothing, c^\varnothing\}$.

Definition 2.4. Convention: $x \in \mathbf{A} \to x \in_\varnothing \mathbf{A}$.

Definition 2.5. Classical Sets:

$$\mathbf{CST}(\mathbf{A}) \iff (\forall \mathbf{x}, \mathbf{s})(\mathbf{x} \in_\mathbf{s} \mathbf{A} \longrightarrow \mathbf{s} = \varnothing \ \&\ \mathbf{x} \textit{ is atomic or } \mathbf{CST}(\mathbf{x})).$$

2.1. Boolean Definitions.

The following definitions preserve familiar CST meanings.

Definition 2.6. Subsets:

$$\mathbf{A} \subseteq \mathbf{B} \longleftrightarrow (\forall x, s)\Big(\ x \in_s \mathbf{A} \ \to\ x \in_s \mathbf{B}\Big),$$
$$\mathbf{A} \subset \mathbf{B} \longleftrightarrow \mathbf{A} \subseteq \mathbf{B} \ \ \&\ \ \mathbf{A} \neq \mathbf{B}.$$

Definition 2.7. Non-Empty Subsets:

$$\mathbf{A} \mathbin{\dot\subset} \mathbf{B} \iff \varnothing \neq \mathbf{A} \subset \mathbf{B},$$
$$\mathbf{A} \mathbin{\dot\subseteq} \mathbf{B} \iff \mathbf{A} \subseteq \mathbf{B} \ \ \&\ \ \mathbf{B} \neq \varnothing \to \mathbf{A} \neq \varnothing.$$

Definition 2.8. Union: $\mathbf{A} \cup \mathbf{B} = \Big\{\ x^y :\ x \in_y \mathbf{A} \ \ \textit{or}\ \ x \in_y \mathbf{B}\ \Big\}$.

EXAMPLE: $\{a^A, b^B, c^C\} \cup \{a^A, c^C, d^D\} = \{a^A, b^B, c^C, d^D\}$.

Definition 2.9. Intersection: $\mathbf{A} \cap \mathbf{B} = \Big\{\ x^y :\ x \in_y \mathbf{A} \ \ \textit{and}\ \ x \in_y \mathbf{B}\ \Big\}$.

EXAMPLE: $\{a^A, b^B, c^C\} \cap \{a^A, c^C, d^D\} = \{a^A, c^C\}$.

Definition 2.10. Relative Complement: $\mathbf{A} \sim \mathbf{B} = \Big\{\ x^y :\ x \in_y \mathbf{A} \ \&\ x \notin_y \mathbf{B}\Big\}$.

EXAMPLE: $\{a^A, b^B, c^C\} \sim \{a^A, c^C, d^D\} = \{b^B\}$.

Definition 2.11. Symmetric Difference:

$$\mathbf{A} \mathbin{\triangle} \mathbf{B} = \Big\{\ x^y :\ (\ x \in_y \mathbf{A} \ \&\ x \notin_y \mathbf{B}\ ) \textit{ or } \ (\ x \in_y \mathbf{B} \ \&\ x \notin_y \mathbf{A}\ )\Big\}.$$

EXAMPLE: $\{a^A, b^B, c^C\} \triangle \{a^A, c^C, d^D\} = \{b^B, d^D\}$.

3. Ordered Pairs & Graphs

A graph is generally considered to be a well-defined relationship expressed by pairs of values on an xy-axis. A graph can be defined as a set containing all pairs of values from a first coordinate with values from a second coordinate. Such a pairing is typically represented by an *ordered pair* notation $<\mathbf{x},\mathbf{y}>$ with $\mathbf{x}$ being a value from the first coordinate and $\mathbf{y}$ being a value from the second coordinate.

Definition 3.1. Graph: $\mathbf{GRAF}(\mathbf{A},\mathbf{B}) \ = \ \{ <a,b>\colon \ a \in \mathbf{A} \ \ \& \ \ b \in \mathbf{B} \ \}$.

Theorem 3.2. $<\mathbf{a},\mathbf{b}> = <\mathbf{x},\mathbf{y}> \ \rightarrow \mathbf{a} = \mathbf{x} \ \& \ \mathbf{b} = \mathbf{y}$.

No set definition is being given to the ordered pair, $<\mathbf{x},\mathbf{y}>$, since it is only intended here to be used for the following CST definitions.

4. CST Graph Operations

Definition 4.1. CST Domain: $\mathbf{DM}(\mathbf{Q}) \ = \ \{ \ x\colon \ (\exists y)(<x,y> \ \in \mathbf{Q})\}$.

EXAMPLE: $\mathbf{DM}(\mathbf{GRAF}(\mathbf{A},\mathbf{B})) \ = \ \mathbf{A}$.

Definition 4.2. CST Range: $\mathbf{RN}(\mathbf{Q}) \ = \ \{ \ y\colon \ (\exists x)(<x,y> \ \in \mathbf{Q})\}$.

EXAMPLE: $\mathbf{RN}(\mathbf{GRAF}(\mathbf{A},\mathbf{B})) \ = \ \mathbf{B}$.

Definition 4.3. CST Restriction: $\mathbf{RS}(\mathbf{Q},\mathbf{A}) \ = \ \{ \ <x,y>\colon <x,y> \ \in \mathbf{Q} \ \ \& \ \ x \in \mathbf{A} \ \}$.

Definition 4.4. CST Image: $\mathbf{IM}(\mathbf{Q},\mathbf{A}) \ = \ \{ \ y : <x,y> \ \in \mathbf{Q} \ \ \& \ \ x \in \mathbf{A} \ \}$.

Note: Image could alternatively be defined by: $\mathbf{IM}(\mathbf{Q},\mathbf{A}) = \mathbf{RN}(\mathbf{RS}(\mathbf{Q},\mathbf{A}))$.

Definition 4.5. Relative Product:

$$\mathbf{RP}(\mathbf{A},\mathbf{B}) \ = \ \{ \ <x,z>\colon \ (\exists y)(<x,y> \ \in \mathbf{A} \ \ \& \ \ <y,z> \ \in \mathbf{B} \ )\}.$$

Noticeably absent from these definitions is the *Cartesian product*, usually required for defining relations and functions. The definition of **GRAF** can be used instead.

5. CST Functions

CST functions are generally defined as graphs with restrictions on allowable ordered pairs. CST functions from domain A to range B can be defined as subsets of $\mathbf{GRAF}(\mathbf{A},\mathbf{B})$.

Definition 5.1. CST Function: For $\mathbf{f} \subseteq \mathbf{GRAF}(\mathbf{A},\mathbf{B})$,

$$\mathbf{f} \ \text{ is a function } \iff (\forall x)\Big(x \in \mathbf{A} \ \ \& \ \mathbf{IM}(\mathbf{f},\{\mathbf{x}\}) \subseteq \mathbf{B} \ \rightarrow \ Sing\big(\mathbf{IM}(\mathbf{f},\{\mathbf{x}\})\big)\Big).$$

With the CST version of *Cartesian product* as a binary operation, it becomes a temptation to ambiguously generate nested ordered pairs to represent sets of n-tuples. The XST version of a Cartesian product allows sets of n-tuples to be unambiguously generated from other sets of tuples, or tuple-sets.

6. Tuple-Set Preliminaries

Though the concept of a *graph* is usually restricted to be just a two dimensional relationship, there seems to be no compelling reason not to define unary, binary, ternary, and n-ary relationships. Using an XST defined concept of an n-tuple, n-ary relationships can easily be defined.

Definition 6.1. n-tuple: $<x_1,x_2,..,x_n> = \ \{x_1^1, x_2^2, .., x_n^n\}$.

This definition needs more refinement to preclude duplicate scope values, allow missing scope values and qualify element values. As it stands $< x >$, $< x, y >$, $< x, y, z >$, and $< x, < x, y >>$ are well-defined. A unary relationship could be defined as a set of 1-tuples. A ternary relationship as a set of 3-tuples And an n-ary relationship as a set of n-tuples. Some CST set operations already exist for binary relationships, or sets of ordered pairs.

Tuple-sets, or sets of n-tuples, are unlike other sets in that the integer scopes on elements allow for an interpretation of *structure* on, or *ordering* of, elements. This allows special set operations to be defined that exploit the presence of integer scopes.

Exploiting scopes requires two heavily used operations needed to compare and manipulate scope values. These two operations are extremely important, but equally unfriendly.

Definition 6.2. Re-Scope by Scope:

$$\mathbf{z}^{/\sigma/} \;=\; \big\{\mathbf{x}^{w}\colon\; (\exists s)\big(\;\mathbf{x} \in_s \mathbf{z} \;\;\&\;\; s \in_w \sigma\;\big)\big\}. \quad \textit{example}\; \{..,\mathbf{x}^s,..\}^{/\{..,s^w,..\}/} \;=\; \{..,\mathbf{x}^w,..\}$$

Definition 6.3. Re-Scope by Element:

$$\mathbf{z}^{\backslash\sigma\backslash} \;=\; \big\{\mathbf{x}^{\mathbf{s}}\colon\; (\exists w)\big(\;\mathbf{x} \in_w \mathbf{z} \;\;\&\;\; s \in_w \sigma\;\big)\big\}. \quad \textit{example}\; \{..,\mathbf{x}^w,..\}^{\backslash\{..,s^w,..\}\backslash} \;=\; \{..,\mathbf{x}^s,..\}$$

These two definitions are duals. They both replace scope values of a given set. One with scope values from another set. The other with element values from another set.

For example:

$$\{x^a, y^b, z^d\}^{/\{a^A,b^B,d^D\}/} = \{x^A, y^B, z^D\},$$
$$\{x^A, y^B, z^D\}^{\backslash\{a^A,b^B,d^D\}\backslash} = \{x^a, y^b, z^d\}.$$

Definition 6.4. Tuple Maximum Scope Value: $tup(x) = n \iff x = \{\; x_1^1, x_2^2, .., x_n^n\;\}$.

Definition 6.5. Tuple Concatenation:

For $tup(x) = j,\; tup(y) = k,\; \&\; \sigma =< j+1, , j+k >,\; x \cdot y \;=\; x \cup y^{\backslash\sigma\backslash}$

EXAMPLE: $\langle a, b, c, d\rangle \cdot \langle w, x, y, z\rangle \;=\; \langle a, b, c, d, w, x, y, z\rangle.$

Note: $tup(x) = n \;\;\&\;\; tup(y) = m \;\longrightarrow\; tup(x \cdot y) = n + m.$

Definition 6.6. $\varnothing \cdot x \;=\; x \cdot \varnothing \;=\; x.$

Theorem 6.7. $x \cdot y \cdot z \;=\; x \cdot (y \cdot z) \;=\; (x \cdot y) \cdot z.$

It should be emphasized that in XST tuples are not an ambiguous notational expediency, nor an alias for nested ordered pairs, as is the case in CST.[2] In XST a tuple reflects a unique underlying membership condition belonging to a specific class of sets having particularly interesting membership properties.

7. XST Tuple-Set Operations

All CST Graph operations are defined in XST using tuple-sets. Elements of CST sets are replaced by 1-tuples. For example the Domain of set $\{< a, b >, < x, y >\}$ of ordered pairs, is $\{< a >, < x >\}$ and the Range is $\{< b >, < y >\}$

Carrying this theme to the extreme, the i-th Domain of a set of n-tuples would be the set of i-th values from all n-tuples expressed as a set of 1-tuples. CST Range becomes the 2nd Domain of a set of ordered pairs.

Definition 7.1. Domain Extraction:

$$\mathbf{Đ}_\sigma(\mathbf{Q}) = \Big\{\;\mathbf{x}^s\colon (\exists \mathbf{z}, w)\big(\;\mathbf{z} \in_w \mathbf{Q} \;\;\&\; \mathbf{x} = \mathbf{z}^{/\sigma/} \neq \varnothing \;\;\&\; s = w^{/\sigma/}\;\big)\Big\}.$$

For example:

$$\mathbf{Đ}_{<1>}(< a, x >, < b, y >, < c, z >) = \{< a >, < b >, < c >\},$$
$$\mathbf{Đ}_{<2>}(< a, x >, < b, y >, < c, z >) = \{< x >, < y >, < z >\},$$
$$\mathbf{Đ}_{<4,5,1>}(< a, b, c, e, f >, < w, v, x, y, z >) = \{< e, f, a >, < y, z, w >).$$

The behavior of the CST Range operation is absorbed by the behavior of the XST Domain operation as influenced by the value of σ. It will be shown that, when conditions are right and $\sigma = <2>$, the result of a XST Domain operation is compatible with CST Range.

Definition 7.2. Restriction:

$$\mathbf{Q}\,\underset{\sigma}{|}\,\mathbf{A} = \Big\{ \mathbf{z}^{w} \colon \big(\mathbf{z} \in_{w} \mathbf{Q}\big) \ \&\ \ (\exists\, a, s)\big(a \in_{s} \mathbf{A} \ \&\ a^{\backslash\sigma\backslash} \subseteq\!\!\!\!\cdot\ \mathbf{z} \ \ \&\ s^{\backslash\sigma\backslash} \subseteq w\big)\Big\}.$$

$\mathbf{Q}\,\underset{\sigma}{|}\,\mathbf{A}$ yields a subset of $\mathbf{Q}$ *restricted by* $\mathbf{A}$ under the influence of σ. This provides a wide variety for functionally selecting related items from elements of the set $\mathbf{Q}$.

Definition 7.3. Image: $\mathbf{Q}\big[\mathbf{A}\big]_{<\sigma_1,\sigma_2>} = \text{Đ}_{\sigma_2}(\mathbf{Q}\,\underset{\sigma_1}{|}\,\mathbf{A})$.

Definition 7.4. Image:(alternate definition)

$$\begin{aligned}\mathbf{Q}\big[\mathbf{A}\big]_{<\sigma_1,\sigma_2>} = \Big\{ \mathbf{x}^{\mathbf{s}} \colon & (\exists\, z, w, a, t)\big(\ z \in_{w} \mathbf{Q}\big) \ \ \&\ \ \big(a \in_{t} \mathbf{A}\big) \ \ \& \\ & 1)\ \ \big(a^{\backslash\sigma_1\backslash} \subseteq\!\!\!\!\cdot\ z \ \ \&\ t^{\backslash\sigma_1\backslash} \subseteq w\big) \ \ \& \\ & 2)\ \ \big(\mathbf{x} = z^{/\sigma_2/} \neq \varnothing \ \ \&\ \mathbf{s} = w^{/\sigma_2/}\ \big)\Big\}.\end{aligned}$$

[Where '$\subseteq\!\!\!\!\cdot$' means non-empty subset.]

Definition 7.5. Convention: $\mathbf{Q}\big[\mathbf{A}\big] = \mathbf{Q}\big[\mathbf{A}\big]_{<<1>,<2>>}$.

With the CST version of *Cartesian product* as a binary operation, it becomes a temptation to ambiguously generate nested ordered pairs to represent sets of n-tuples. The XST version of a Cartesian product allows sets of tuples to be unambiguously generated from other sets of tuples, or tuple-sets.

Definition 7.6. XST Cross Product: $\mathbf{A} \otimes \mathbf{B} = \{\ \mathbf{x} \cdot \mathbf{y} :\ \mathbf{x} \in \mathbf{A}\ \&\ \mathbf{y} \in \mathbf{B}\ \}$.
Note: Both $\mathbf{x}$ and $\mathbf{y}$ are tuples.

Theorem 7.7. $\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C} \ =\ \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) \ =\ (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$.

Let a **Unary Set** be defined by $\overline{\mathbf{A}} \ =\ \{\ <x>: x \in \mathbf{A}\ \}$, then in XST terms

$$\mathbf{GRAF}(\mathbf{A},\mathbf{B}) \ =\ \{\ x \cdot y : <x> \in \overline{\mathbf{A}} \ \ \& <y> \in \overline{\mathbf{A}}\ \} \ =\ \overline{\mathbf{A}} \otimes \overline{\mathbf{B}}.$$

8. Relative Product

Relative Product has more personality than other operations in that given the same two operands the resultant set can have many forms. Though, in CST the operation is rather bland matching the range elements of the first operand with the domain elements of the second operand and producing a pair of the domain element of the first with the range element of the second.

For example, in CST: $\{<a,b>\}/\{<b,c>\} = \{<a,c>\}$. Following are some element combinations that are potentially more interesting:

1) $<a,b>\ \ \& <b,c> \longrightarrow <a,c>$,
2) $<a,b>\ \ \& <b,c> \longrightarrow <a,b,c>$,
3) $<a,b>\ \ \& <a,c> \longrightarrow <a,b,c>$,
4) $<a,b>\ \ \& <a,c> \longrightarrow <b,c>$,
5) $<a,c>\ \ \& <b,c> \longrightarrow <a,b,c>$,
6) $<a,c>\ \ \& <b,c> \longrightarrow <a,b>$,
7) $<a,b,c>\ \ \& <x,y,c,b> \longrightarrow <b,c,a,y,b,c,x,x>$,
8) $<a,b,c>\ \ \& <x,y,z> \longrightarrow <a,b,c,x,y,z>$.

All of the above are producible with the following definition.

Definition 8.1. Relative Product:

$$\mathbf{F}\Big/_{<\sigma_1,\sigma_2>}^{<\omega_1,\omega_2>}\mathbf{G} = \Big\{\mathbf{z}^{\tau} \colon (\exists x, s, y, t)(\ x \in_s \mathbf{F} \ \&\ y \in_t \mathbf{G} \ \ \&\ \ x^{/\sigma_2/} = y^{/\omega_1/} \ \ \& \\ s^{/\sigma_2/} = t^{/\omega_1/} \ \ \&\ \ \mathbf{z} = x^{/\sigma_1/} \cup y^{/\omega_2/} \ \ \&\ \ \tau = s^{/\sigma_1/} \cup t^{/\omega_2/})\ \}.$$

For $\mathbf{f}\Big/_{\sigma}^{\omega}\mathbf{g}$ using element combinations (1)-(8) above, the following values for σ and ω support the corresponding mappings above.

1) $\sigma =< \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega =< \{1^1\},\ \{2^2\} >,$
2) $\sigma =< \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega =< \{1^1\},\ \{1^2, 2^3\} >,$
3) $\sigma =< \{1^1, 2^2\},\ \{1^1\} > \ \ \&\ \ \omega =< \{1^1\},\ \{2^3\} >,$
4) $\sigma =< \{2^1\},\ \{1^1\} > \ \ \&\ \ \omega =< \{1^1\},\ \{2^2\} >,$
5) $\sigma =< \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega =< \{2^1\},\ \{1^2, 2^3\} >,$
6) $\sigma =< \{1^1\},\ \{2^1\} > \ \ \&\ \ \omega =< \{2^1\},\ \{1^2\} >,$
7) $\sigma =< \{2^1, 3^2, 1^3\}, \{2^1, 3^2\} > \ \ \&\ \ \omega =< \{4^1, 3^2\}\{2^4, 4^5, 3^6, 1^7, 1^8\} >,$
8) $\sigma =< \{1^1, 2^2, 3^3\},\ \varnothing > \ \ \&\ \ \omega =< \varnothing,\ \{4^1, 5^2, 6^3\} > .$

9. XST Functions (as processes)

The term **function** seems to suggest an action, behavior, or process capable of producing a result. Given a definition of **process**, a function could be considered a special process.[3]

Definition 9.1. Process: Two sets $\mathbf{f}$ and σ define a process $\mathbf{f}_{(\sigma)}$ iff

$$(\exists x)\big(\ \mathbf{f}_{(\sigma)}(x) \neq \varnothing\big)\ \&\ (\forall \mathbf{g})(\mathbf{g} \subseteq \mathbf{f})(\exists x)\big(\ \mathbf{g}_{(\sigma)}(x) \neq \varnothing\ \big).$$

[Where '$\subseteq$' means non-empty subset.]

The above definition only asserts that a process is an abstraction somehow related to an interaction of sets. No demands are made on how such a relationship need be supported. The important property of a process is that it is not a set. '$\mathbf{f}_{(\sigma)}(x) \in_s \mathbf{Q}$' makes mathematical sense, while '$\mathbf{f}_{(\sigma)} \in_s \mathbf{Q}$' does not. The expression '$\mathbf{f}_{(\sigma)}(x)$' defines a set-membership condition, while the expression '$\mathbf{f}_{(\sigma)}$' defines a set-behavior.

Definition 9.2. Process Equality:

$$\mathbf{f}_{(\sigma)} = \mathbf{g}_{(\omega)} \iff (\forall x)(\ \mathbf{f}_{(\sigma)}(x) = \mathbf{g}_{(\omega)}(x)\ ).$$

All uses of processes are only a prediction, the actual set behavior can not be realized until a process is given a set-theoretic interpretation.

10. Applications

Given that extended image operation, $\mathbf{f}\big[x\big]_{\sigma}$, has a well defined set-theoretic definition, the following definitions of *application* transform a process into a set-theoretic reality. Since a process could be applied to a set or to a process, two definitions of application are required.

Definition 10.1. Application(set): $\mathbf{f}_{(\sigma)}(x) \ = \ \mathbf{f}\big[x\big]_{\sigma}.$

A process is a function if the image of every singleton produces a singleton set.

Definition 10.2. Function:

$$\mathbf{f}_{(\sigma)} \ \text{ is a function } \iff (\forall y)\Big(\ \mathit{Sing}(y) \ \ \&\ \ \mathbf{f}\big[y\big]_{\sigma} \neq \varnothing \ \rightarrow \ \ \mathit{Sing}\Big(\mathbf{f}\big[y\big]_{\sigma}\Big)\ \Big).$$

Note: This definition of function does not require nor limit a function to being a graph.

Definition 10.3. Application(process): $\mathbf{f}_{(\sigma)}(\mathbf{g}_{(\omega)}) = \Big(\mathbf{f}_{(\sigma)}(\mathbf{g})\Big)_{(\omega)} = \Big(\mathbf{f}\big[\mathbf{g}\big]_{\sigma}\Big)_{(\omega)}.$

A process applied to a set produces a set. When applied to a process produces a process. Nested applications, self application, Lambda application and the composition of multiple functions is presented in [4].

References

[1] Blass, A.; Childs, D L: *Axioms and Models for an Extended Set Theory*, A Formal Foundation for Unified Modeling of Mathematical Objects: - June 2011
http://www.math.lsa.umich.edu/%7Eablass/XST_Axioms.pdf

[2] Skolem, T.: *Two remarks on set theory,*(n-tuples as sets), *Math. Scand.* 5 (1957) 43–46.
http://xsp.xegesis.org/T_Skolem.pdf

[3] Childs, D L: *Functions As Set Behavior*, Essential Concepts - 2018
http://xegesis.org/xsp/Process.pdf

[4] Childs, D L: *Processes, Functions, Applications & Composition*, Summary - 2018
http://xegesis.org/xsp/PFAC.pdf

Appendix A. PROPERTIES

A.1. **Preserved CST Properties.**

The usual CST properties associated with the Domain operation are preserved in XST.

Consequences A.1. *Preserved Domain Properties:*

$(a)\ \mathbf{Đ}_\sigma(\mathbf{Q} \cup \mathbf{R}) \ = \ \mathbf{Đ}_\sigma(\mathbf{Q}) \cup \mathbf{Đ}_\sigma(\mathbf{R}),$

$(b)\ \mathbf{Đ}_\sigma(\mathbf{Q} \cap \mathbf{R}) \ \subseteq \ \mathbf{Đ}_\sigma(\mathbf{Q}) \cap \mathbf{Đ}_\sigma(\mathbf{R}),$

$(c)\ \mathbf{Đ}_\sigma(\mathbf{Q}) \sim \mathbf{Đ}_\sigma(\mathbf{R}) \ \subseteq \ \mathbf{Đ}_\sigma(\mathbf{Q} \sim \mathbf{R}),$

$(d)\ \mathbf{Q} \mathrel{\dot\subseteq} \mathbf{R} \ \longrightarrow \ \mathbf{Đ}_\sigma(\mathbf{Q}) \subseteq \mathbf{Đ}_\sigma(\mathbf{R}),$

$(e)\ \mathbf{Đ}_\varnothing(\mathbf{Q}) = \varnothing.$

The usual CST properties associated with set Restriction are preserved in XST.

Consequences A.2. *Preserved Set Restriction Properties:*

$(a)\ \mathbf{Q}\big|_\sigma(\mathbf{A} \cup \mathbf{B}) \ = \ \mathbf{Q}\big|_\sigma\mathbf{A} \cup \mathbf{Q}\big|_\sigma\mathbf{B},$

$(b)\ \mathbf{Q}\big|_\sigma(\mathbf{A} \cap \mathbf{B}) \ \subseteq \ \mathbf{Q}\big|_\sigma\mathbf{A} \cap \mathbf{Q}\big|_\sigma\mathbf{B},$

$(c)\ \mathbf{Q}\big|_\sigma\mathbf{A} \ \sim \ \mathbf{Q}\big|_\sigma\mathbf{B} \ \subseteq \ \mathbf{Q}\big|_\sigma(\mathbf{A} \sim \mathbf{B}),$

$(d)\ \mathbf{A} \mathrel{\dot\subseteq} \mathbf{B} \ \longrightarrow \ \mathbf{Q}\big|_\sigma\mathbf{A} \mathrel{\dot\subseteq} \mathbf{Q}\big|_\sigma\mathbf{B},$

$(e)\ \mathbf{Q}\big|_\sigma(\mathbf{Đ}_\sigma(\mathbf{Q}) \cap \mathbf{A}) \ = \ \mathbf{Q}\big|_\sigma\mathbf{A},$

$(f)\ \mathbf{Q}\big|_\sigma\mathbf{A} \ \subseteq \ \mathbf{Q},$

$(g)\ \mathbf{Q}\big|_\sigma\varnothing = \varnothing \ \ \& \ \ \mathbf{Q}\big|_\varnothing\mathbf{A} = \varnothing,$

$(h)\ \mathbf{Đ}_\sigma(\mathbf{Q}) \cap \mathbf{A} = \varnothing \ \longrightarrow \ \mathbf{Q}\big|_\sigma\mathbf{A} = \varnothing.$

$(i)\ (\mathbf{Q} \cup \mathbf{R})\big|_\sigma\mathbf{A} \ = \ \mathbf{Q}\big|_\sigma\mathbf{A} \cup \mathbf{R}\big|_\sigma\mathbf{A},$

$(j)\ (\mathbf{Q} \cap \mathbf{R})\big|_\sigma\mathbf{A} \ \subseteq \ \mathbf{Q}\big|_\sigma\mathbf{A} \cap \mathbf{R}\big|_\sigma\mathbf{A},$

$(k)\ \mathbf{Q}\big|_\sigma\mathbf{A} \sim \mathbf{R}\big|_\sigma\mathbf{A} \ \subseteq \ (\mathbf{Q} \sim \mathbf{R})\big|_\sigma\mathbf{A},$

$(l)\ \mathbf{Q} \mathrel{\dot\subseteq} \mathbf{R} \ \longrightarrow \ \mathbf{Q}\big|_\sigma\mathbf{A} \subseteq \mathbf{R}\big|_\sigma\mathbf{A},$

The usual CST properties associated with the Image operation are preserved in XST.

Consequences A.3. *Preserved Image Properties:*

$(a)\ \mathbf{Q}\big[\mathbf{A} \cup \mathbf{B}\big]_\sigma \ = \ \mathbf{Q}\big[\mathbf{A}\big]_\sigma \cup \mathbf{Q}\big[\mathbf{B}\big]_\sigma,$

$(b)\ \mathbf{Q}\big[\mathbf{A} \cap \mathbf{B}\big]_\sigma \ \subseteq \ \mathbf{Q}\big[\mathbf{A}\big]_\sigma \cap \mathbf{Q}\big[\mathbf{B}\big]_\sigma,$

$(c)\ \mathbf{Q}\big[\mathbf{A}\big]_\sigma \sim \mathbf{Q}\big[\mathbf{B}\big]_\sigma \ \subseteq \ \mathbf{Q}\big[\mathbf{A} \sim \mathbf{B}\big]_\sigma,$

(d) $\mathbf{A} \mathrel{\underset{\cdot}{\subseteq}} \mathbf{B} \longrightarrow \mathbf{Q}\big[\mathbf{A}\big]_{\sigma} \subseteq \mathbf{Q}\big[\mathbf{B}\big]_{\sigma}$,

(e) $\mathbf{Q}\big[\boldsymbol{Đ}_{\sigma}(\mathbf{Q}) \cap \mathbf{A}\big]_{<\sigma,\gamma>} = \mathbf{Q}\big[\mathbf{A}\big]_{<\sigma,\gamma>}$,

(f) $\mathbf{Q}\big[\mathbf{A}\big]_{<\sigma,\gamma>} = \boldsymbol{Đ}_{\gamma}(\mathbf{Q}\,|_{\sigma}\,\mathbf{A})$.

(g) $\mathbf{Q}\big[\varnothing\big]_{\sigma} = \varnothing, \quad \varnothing\big[\mathbf{A}\big]_{\sigma} = \varnothing, \quad \mathbf{Q}\big[\mathbf{A}\big]_{\varnothing} = \varnothing$,

(h) $\boldsymbol{Đ}_{\sigma}(\mathbf{Q}) \cap \mathbf{A} = \varnothing \longrightarrow \mathbf{Q}\big[\mathbf{A}\big]_{<\sigma,\gamma>} = \varnothing$,

(i) $(\mathbf{Q} \cup \mathbf{R})\big[\mathbf{A}\big]_{\sigma} = \mathbf{Q}\big[\mathbf{A}\big]_{\sigma} \cup \mathbf{R}\big[\mathbf{A}\big]_{\sigma}$,

(j) $(\mathbf{Q} \cap \mathbf{R})\big[\mathbf{A}\big]_{\sigma} \subseteq \mathbf{Q}\big[\mathbf{A}\big]_{\sigma} \cap \mathbf{R}\big[\mathbf{A}\big]_{\sigma}$,

(k) $\mathbf{Q}\big[\mathbf{A}\big]_{\sigma} \sim \mathbf{R}\big[\mathbf{A}\big]_{\sigma} \subseteq (\mathbf{Q} \sim \mathbf{R})\big[\mathbf{A}\big]_{\sigma}$

(l) $\mathbf{Q} \mathrel{\underset{\cdot}{\subseteq}} \mathbf{R} \longrightarrow \mathbf{Q}\big[\mathbf{A}\big]_{\sigma} \subseteq \mathbf{R}\big[\mathbf{A}\big]_{\sigma}$.